

Heat kernel approach for sup-norm bounds for cusp forms of integral and half-integral weight

Anilatmaja Aryasomayajula

Abstract

In this article, using the heat kernel approach from [2], we derive sup-norm bounds for cusp forms of integral and half-integral weight. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a cocompact Fuchsian subgroup of first kind. For $k \in \frac{1}{2}\mathbb{Z}$ (or $k \in 2\mathbb{Z}$), let $S^k(\Gamma)$ denote the complex vector space of weight- k cusp forms. Let $\{f_1, \dots, f_{j_k}\}$ denote an orthonormal basis of $S^k(\Gamma)$. In this article, we show that as $k \rightarrow \infty$, the sup-norm for $\sum_{i=1}^{j_k} y^k |f_i(z)|^2$ is bounded by $O(k)$, where the implied constant is independent on Γ . Furthermore, using results from [1], we extend these results to the case when Γ is cofinite.

Mathematics Subject Classification (2010): 30F30, 30F35, 30F45.

1 Introduction

The main aim of this article is to demonstrate the usefulness of the heat kernel approach to study automorphic forms. J. Jorgenson and J. Kramer have successfully applied heat kernel analysis to study automorphic forms, and derived optimal estimates for various arithmetic invariants. We use a slightly different approach, which is more geometric and derive similar estimates for integral weight cusp forms as in [4]. We extend these estimates to half-integral weight cusp forms.

Notation Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a cocompact Fuchsian subgroup of first kind acting on the hyperbolic upper half-plane \mathbb{H} , via fractional linear transformations. Let X denote the quotient space $\Gamma \backslash \mathbb{H}$, and let g denote the genus of X . We assume that $g > 1$.

We assume that Γ has no elliptic elements, so the quotient space X admits the structure of a Riemann surface.

We denote the $(1, 1)$ -form corresponding to the hyperbolic metric of X , which is compatible with the complex structure on X and has constant negative curvature equal to minus one, by $\mu_{\mathrm{hyp}}(z)$. Locally, for $z \in X$, it is given by

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{\mathrm{Im}(z)^2}.$$

Let $\mu_{\mathrm{shyp}}(z)$ denote the rescaled hyperbolic metric $\mu_{\mathrm{hyp}}(z)/\mathrm{vol}_{\mathrm{hyp}}(X)$, which measures the volume of X to be one.

For $k \in \frac{1}{2}\mathbb{Z}$ (or $2\mathbb{Z}$), let $S^k(\Gamma)$ denote the complex vector space of weight- k cusp forms. Let j_k denote the dimension of $S^k(\Gamma)$, and let $\{f_1, \dots, f_{j_k}\}$ denote an orthonormal basis

of $S^k(\Gamma)$ with respect to the Petersson inner product. Put

$$\mathcal{B}_X^k(z) := \sum_{i=1}^{j_k} y^k |f_i(z)|^2.$$

With notation as above, for $k \in \frac{1}{2}\mathbb{Z}$ (or $2\mathbb{Z}$), we have the following estimate

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) = O(1), \quad (1)$$

where the implied constant is independent of Γ .

Let Γ now be a cofinite Fuchsian subgroup without elliptic elements, which implies that $X = \Gamma \backslash \mathbb{H}$ admits the structure of a noncompact Riemann surface of finite volume. Then, for any $z \in X$ and $k \in \frac{1}{2}\mathbb{Z}$, we have

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) \leq \frac{1}{8\pi}; \quad (2)$$

and for $k \in 2\mathbb{Z}$, we have

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) \leq \frac{1}{2\pi}. \quad (3)$$

Applications and existing results When X is comcompact, in proving estimate (1), for any $z \in X$ and $k \in \frac{1}{2}\mathbb{Z}$, we first arrive at the following estimate

$$\lim_k \frac{1}{k} \mathcal{B}_X^k(z) = O(1),$$

where the implied constant is independent of Γ . So, let

$$\lim_k \frac{1}{k} \mathcal{B}_X^k(z) = C.$$

Hence, for $z \in X$ and $k \in \frac{1}{2}\mathbb{Z}$ (or $2\mathbb{Z}$) and f any smooth function on X , we observe that

$$\lim_k \frac{1}{Ck} \int_X f(z) \mathcal{B}_X^k(z) \mu_{\text{shyp}}(z) = \int_X f(z) \mu_{\text{shyp}}(z),$$

which proves a version of arithmetic quantum unique ergodicity conjecture in the average case for half-integral weight cusp forms.

For $k \in \frac{1}{2}\mathbb{Z}$ and $N \in \mathbb{N}$, let f be any weight- k cusp form with respect to the arithmetic subgroup $\Gamma_0(4N)$. Furthermore, let f be normalized with respect to the Petersson inner-product. Then, in [3], Kiral has derived the following estimate

$$\sup_{z \in Y_0(N)} y^k |f(z)|^2 = O_{k,\varepsilon}(N^{\frac{1}{2} - \frac{1}{18} + \varepsilon}),$$

for any $\varepsilon > 0$. Using the above estimate, one can derive

$$\sup_{z \in Y_0(N)} \mathcal{B}_{Y_0(N)}(z) = O_{k,\varepsilon}(N^{\frac{1}{2} - \frac{1}{18} + \varepsilon}),$$

for any $\varepsilon > 0$.

For $k \in 2\mathbb{Z}$, Friedman, Jorgenson, and Kramer have derived the estimates $O(k)$ and $O(k^{3/2})$ for $\mathcal{B}_X^k(z)$ in the cocompact and cofinite cases, respectively in [4]. Extending their heat kernel techniques from [5] to higher weights, they showed that the implied constant is independent of the Fuchsian subgroup Γ .

For $k \in \frac{1}{2}\mathbb{Z}$, estimate (1) shows the dependence of \mathcal{B}_X^k on k , and our result holds true for any arbitrary cocompact Fuchsian group.

For $k \in 2\mathbb{Z}$, estimate (1) gives an alternate proof for the estimate of Jorgenson and Kramer in the cocompact case. Our method is more geometric and relies on a theorem of Bouche in [2], which is based on the micro-local analysis of the heat kernel.

Estimate (1) extends with notational changes to cusp forms of integral and half-integral weight associated to cocompact Hilbert modular groups and Siegel modular groups.

It is possible to extend estimate (1) to cofinite Fuchsian subgroups by studying Bouche's methods in a cuspidal neighborhood.

2 Heat kernels on compact complex manifolds

In this section, we recall the main results from [2] and [1], which we use in the next section.

Let (M, ω) be a compact complex manifold of dimension n with a Hermitian metric ω . Let \mathcal{L} be a positive Hermitian holomorphic line bundle on M with the Hermitian metric given by $\|s(z)\|_{\mathcal{L}}^2 := e^{-\phi(z)}|s(z)|^2$, where $s \in \mathcal{L}$ is any section, and $\phi(z)$ is a real-valued function defined on M .

For any $k \in \mathbb{N}$, let $\bar{\square}_k := (\bar{\partial}^* + \bar{\partial})^2$ denote the $\bar{\partial}$ -Laplacian acting on smooth sections of the line bundle $\mathcal{L}^{\otimes k}$. Let $K_{M, \mathcal{L}}^k(t; z, w)$ denote the smooth kernel of the operator $e^{-\frac{2t}{k} \bar{\square}_k}$. We refer the reader to p. 2 in [2], for the details regarding the properties which uniquely characterize the heat kernel $K_{M, \mathcal{L}}^k(t; z, w)$. When $z = w$, the heat kernel $K_{M, \mathcal{L}}^k(t; z, w)$ admits the following spectral expansion

$$K_{M, \mathcal{L}}^k(t; z, w) = \sum_{n \geq 0} e^{-\frac{2t}{k} \lambda_n^k} \varphi_n(z) \otimes \varphi_n^*(w), \quad (4)$$

where $\{\lambda_n^k\}_{n \geq 0}$ denotes the set of eigenvalues of $\bar{\square}_k$ (counted with multiplicities), and $\{\varphi_n\}_{n \geq 0}$ denotes a set of associated orthonormal eigenfunctions.

Let $\{s_i\}$ denote an orthonormal basis of $H^0(M, \mathcal{L}^{\otimes k})$. For any $z \in M$, the Bergman kernel is given by

$$\mathcal{B}_{M, \mathcal{L}}^k(z) := \sum_i \|s_i(z)\|_{\mathcal{L}^{\otimes k}}^2. \quad (5)$$

For any $z \in M$ and $t \in \mathbb{R}_{>0}$, from the spectral expansion of the heat kernel $K_{M, \mathcal{L}}^k(t; z, w)$ described in equation (4), it is easy to see that

$$\mathcal{B}_{M, \mathcal{L}}^k(t; z) \leq K_{M, \mathcal{L}}^k(t; z, z) \quad \text{and} \quad \lim_t K_{M, \mathcal{L}}^k(t; z, z) = \mathcal{B}_{M, \mathcal{L}}^k(t; z). \quad (6)$$

For $z \in M$, let $c_1(\mathcal{L})(z) := \frac{i}{2\pi} \partial \bar{\partial} \phi(z)$ denote the first Chern form of the line bundle \mathcal{L} . Let $\alpha_1, \dots, \alpha_n$ denote the eigenvalues of $\partial \bar{\partial} \phi(z)$ at the point $z \in M$. Then, with notation

as above, from Theorem 1.1 in [2], for any $z \in M$ and $t \in (0, k^\varepsilon)$, for a given $\varepsilon > 0$ not depending on k , we have

$$\lim_k \frac{1}{k^n} K_{M, \mathcal{L}}^k(t; z, z) = \prod_{j=1}^n \frac{\alpha_j}{(4\pi)^n \sinh(\alpha_j t)}, \quad (7)$$

and the convergence of the above limit is uniform in z .

Using equations (6) and (7), in Theorem 2.1 in [2], Bouche derived the following asymptotic estimate

$$\lim_k \frac{1}{k^n} \mathcal{B}_{M, \mathcal{L}}^k(z) = O(\det_\omega(c_1(\mathcal{L})(z))), \quad (8)$$

and the convergence of the above limit is uniform in $z \in X$.

When M is a noncompact complex manifold, using micro-local analysis of the Bergman kernel, in [1], Berman derived the following estimate

$$\limsup_k \frac{1}{k^n} \mathcal{B}_{M, \mathcal{L}}^k(z) \leq \det_\omega(c_1(\mathcal{L})(z)). \quad (9)$$

3 Estimates of cusp forms

Let notation be as in Section 1. We allow Γ to be cofinite, i.e. X can be a noncompact Riemann surface of finite volume. Let Ω_X denote the cotangent bundle over X . Then, for any $k \in 2\mathbb{Z}$, cusp forms of weight k with respect to Γ are global section of the line bundle $\Omega_X^{\otimes k/2}$. Furthermore, recall that for any $f \in \Omega_X$, i.e., f a weight-2 cusp form, the Petersson metric on the line bundle Ω_X is given by

$$\|f(z)\|_{\Omega_X}^2 := y^2 |f(z)|^2. \quad (10)$$

Let ω_X denote the line bundle of cusp forms of weight $\frac{1}{2}$ over X . Then, for any $k \in \frac{1}{2}\mathbb{Z}$, cusp forms of weight- k with respect to Γ are global section of the line bundle $\omega_X^{\otimes 2k}$. Furthermore, recall that for any $f \in \omega_X$, i.e., f a weight- $\frac{1}{2}$ cusp form, the Petersson metric on the line bundle ω_X is given by

$$\|f(z)\|_{\omega_X}^2 := y^{1/2} |f(z)|^2. \quad (11)$$

Remark 1. For any $z \in X$ and $k \in 2\mathbb{Z}$, from the definition of the Bergman kernel $\mathcal{B}_{X, \Omega_X}^{k/2}(z)$ for the line bundle $\Omega_X^{\otimes k/2}$ from equation (5), we have

$$\mathcal{B}_{X, \Omega_X}^{k/2}(z) = \mathcal{B}_X^k(z).$$

Similarly, for any $z \in X$ and $k \in \frac{1}{2}\mathbb{Z}$, from the definition of the Bergman kernel $\mathcal{B}_{X, \omega_X}^{2k}(z)$ for the line bundle $\omega_X^{\otimes 2k}$ from equation (5), we have

$$\mathcal{B}_{X, \omega_X}^{2k}(z) = \mathcal{B}_X^k(z). \quad (12)$$

Theorem 2. *Let notation be as in Section 1, and we assume that Γ is cocompact, i.e. X is compact. Then, for $k \in \frac{1}{2}\mathbb{Z}$ (or $2\mathbb{Z}$), we have the following estimate*

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) = O(1),$$

where the implied constant is independent of Γ .

Proof. We prove the theorem for $k \in \frac{1}{2}\mathbb{Z}$, and the case for $k \in 2\mathbb{Z}$ follows automatically with notational changes. For any $z \in X$, observe that

$$c_1(\omega_X)(z) = -\frac{i}{2\pi} \partial \bar{\partial} \log(y^{1/2} |f(z)|^2),$$

where f is any cusp form of weight $\frac{1}{2}$. From the above equation, we derive

$$c_1(\omega_X)(z) = \frac{i}{16\pi} \cdot \frac{dz \wedge \bar{z}}{y^2} = \frac{1}{8\pi} \mu_{\text{hyp}}(z),$$

which shows that the line bundle ω_X is positive, and $\det_{\mu_{\text{hyp}}}(c_1(\omega_X)(z)) = \frac{1}{8\pi}$. So applying estimate (8) to the complex manifold X with its natural Hermitian metric μ_{hyp} and the line bundle $\omega_X^{\otimes k}$, and using equation (12), we find

$$\lim_k \frac{1}{k} \mathcal{B}_X^k(z) = \lim_k \frac{1}{k} \mathcal{B}_{X, \omega_X}^{2k}(z) = O\left(2 \det_{\mu_{\text{hyp}}}(c_1(\omega_X)(z))\right) = O(1).$$

As the above limit convergences uniformly in $z \in X$, and as X is compact, we have

$$\sup_{z \in X} \lim_k \frac{1}{k} \mathcal{B}_X^k(z) = \lim_k \sup_{z \in X} \frac{1}{k} \mathcal{B}_X^k(z) = O(1),$$

which completes the proof of the theorem. \square

Corollary 3. *Let notation be as in Section 1, and we assume that Γ is cofinite, i.e. X is a noncompact Riemann surface of finite volume. Then, for $z \in X$ and $k \in \frac{1}{2}\mathbb{Z}$, we have*

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) \leq \frac{1}{8\pi};$$

and for $k \in 2\mathbb{Z}$, we have

$$\limsup_k \frac{1}{k} \mathcal{B}_X^k(z) \leq \frac{1}{2\pi}.$$

Proof. The proof of the theorem follows from estimate (9), and from similar arguments as in Theorem 2. \square

Acknowledgements The author would like to thank J. Kramer and J. Jorgenson for introducing him to the area of automorphic forms and heat kernels. The author would like to express his gratitude to T. Bouche for providing him with helpful references, and to Archana S. Morye, for many helpful discussions and remarks.

References

- [1] R. J. Berman, Bergman kernels and local holomorphic Morse inequalities, *Math. Z.* 248, (2004), 325–344.
- [2] T. Bouche, Asymptotic results for Hermitian line bundles over complex manifolds: The heat kernel approach, *Higher-dimensional complex varieties*, 67–81, de Gruyter, Berlin, 1996.
- [3] E. M. Kiral, Bounds on Sup-norms of Half Integral Weight Modular Forms *Acta Arithmetica*, Vol. 165, (2014), 385-399.
- [4] J. Friedman, J. Jorgenson, and J. Kramer, Uniform sup-norm bounds on average for cusp forms of higher weights, *arXiv preprint arXiv:1305.1348* (2013).
- [5] J. Jorgenson and J. Kramer, Bounding the sup-norm of automorphic forms, *GAFA* 14, (2004), 1267–1277.

Anilatmaja Aryasomayajula

anilatmaja@gmail.com

Department of Mathematics,

University of Hyderabad,

Prof. C. R. Rao Road, Gachibowli,

Hyderabad, 500046, India